

# Boundary States for $AdS_2$ Branes in $AdS_3$

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## Abstract

We construct boundary states for the  $AdS_2$  D-branes in  $AdS_3$ . We show that, in the semi-classical limit, the boundary states correctly reproduce geometric configurations of these branes. We use the boundary states to compute the one loop free energy of open string stretched between the branes. The result agrees precisely with the open string computation in hep-th/0106129.

# 1 Introduction

Recently much progress has been made in understanding string theory in  $AdS_3$ , three-dimensional anti-de Sitter space. With a background of NS-NS  $B$ -field, the worldsheet is described by the  $SL(2, R)$  WZW model. The spectrum of the WZW model was proposed in [1], and it was verified by exact computation of the one loop free energy in [2]. This has led to a proof of the no ghost theorem for string in  $AdS_3$ , which had been an outstanding problem for more than 10 years.<sup>1</sup> The unitary string spectrum emerged from these results agrees with expectations from the  $AdS/CFT$  correspondence [3, 4] applied to the case of  $AdS_3$  [5, 6, 7]. Correlation functions of vertex operators on the string worldsheet were derived in [8, 9, 10]. In [11], they are used to compute target space correlation functions of the string theory and the results were compared with what we expect for the conformal field theory on the boundary of  $AdS_3$ . The correlation functions turned out to have various singularities, and physical interpretations were given for all of them. It was also shown that the four point functions of the target space conformal field theory obey the factorization rules when they should.

In this paper, we study D-branes whose worldvolume fill  $AdS_2$  subspaces of  $AdS_3$  [12, 13]. We call such D-branes as  $AdS_2$  branes. From the point of view of the CFT dual, these configurations are supposed to describe conformal field theories in two dimensions separated by domain walls, each of which preserves at least one Virasoro algebra [14]. The spectrum of open strings ending on the  $AdS_2$  brane was studied in [15].<sup>2</sup> It was shown that, when the brane carries no fundamental string charge, the open string spectrum is exactly equal to the holomorphic square root of the spectrum of closed strings in  $AdS_3$  derived in [1, 2]. It contains short and long strings, and is invariant under spectral flow.

A boundary state [17, 18] is a useful concept in studying D-branes. Although the construction in [18] assumes rationality of conformal field theory, it was shown in [19, 20, 21, 22] that the idea can be applied to the non-rational case of the Liouville field theory. More recently the technique developed in the Liouville theory is applied to the  $AdS_2$  branes [23, 24, 25, 26]. In this paper, we will closely follow the construction in [23, 24]. We will, however, make a different ansatz about one point functions of closed string operators in a disk worldsheet, which leads to different expressions for the boundary states. We show that, in the semi-classical limit, these new boundary states reproduce the geometric configurations of the  $AdS_2$  branes.

Given a pair of boundary states, it is straightforward to compute a partition function on an annulus worldsheet. This gives the spectrum of the conformal field theory on a strip where the boundary conditions on the two sides of the strip are specified by the choice of the boundary states. We can also compute an annulus partition function when the target

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<sup>1</sup>For a list of historical references, see the bibliography in [1].

<sup>2</sup>A related work was also done in [16].

space Euclidean time is periodically identified. An integral of this partition function over the moduli space of worldsheet then gives the one loop free energy at finite temperature. From this, we can read off the physical spectrum of the open string on the  $AdS_2$  brane. We find that the result agrees with the exact open string computation in [15].

This paper is organized as follows. In section 2, we will compute one point functions of closed string operators on a disk with the boundary conditions corresponding to the  $AdS_2$  branes. Following [23, 24] closely but using a different ansatz, we will derive functional relations for the one point functions. The general solution to the relations will be given. In section 3, we will choose particular solutions which agree with the semi-classical limit of the  $AdS_2$  branes. We will use them to compute annulus amplitudes and derive spectral densities of open string states. We will also compute one loop free energy to identify physical states of open strings on the  $AdS_2$  branes. We will close this paper with some discussions on future directions. In Appendix A, we perform some integrals used in this paper. Various coordinate systems for  $AdS_3$  are summarized in Appendix B. We also list useful formulae of the hypergeometric function and the gamma function in Appendix C.

**Note added:** Toward the completion of this manuscript, we were informed of a related work by B. Ponsot, V. Schomerus, and J. Teschner. Their result of the one point functions on the  $AdS_2$  branes agrees with ours shown in (78). We thank them for communicating their results prior to the publication.

## 2 One point functions on a disk

The boundary state can be found by computing the one point functions of closed string operators on a disk worldsheet. Here we will derive them for the  $AdS_2$  brane, following the approach in [23, 24].

### 2.1 Review of closed string in $AdS_3$

In this paper, we will mostly work in Euclidean  $AdS_3$ , which is the three-dimensional hyperbolic space  $H_3^+$  given by one of the two branches of

$$(X^0)^2 - (X^1)^2 - (X^2)^2 - (X_E^3)^2 = R^2, \quad (1)$$

embedded in  $\mathbf{R}^{1,3}$  with the metric

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX_E^3)^2, \quad (2)$$

where  $R$  is the curvature radius of  $H_3^+$ . It is related to  $AdS_3$  with the Lorentzian signature metric by the analytic continuation  $X^3 \rightarrow X_E^3 = -iX^3$ . The Euclidean  $AdS_3$  can also be

realized as a right-coset space  $SL(2, C)/SU(2)$ . Accordingly, the conformal field theory is the  $SL(2, C)/SU(2)$  coset model. In the semi-classical approximation, which is applicable when the level  $k \sim R^2/\alpha'$  of the  $SL(2, C)$  current algebra is large, states in the theory are given by normalizable functions on the target space. The space of such states is decomposed into a sum of principal continuous representations of  $SL(2, C)$  with  $j = \frac{1}{2} + is$  ( $s \in \mathbf{R}_+$ ). It was shown in [27] that the exact Hilbert space of the coset model at finite value of  $k$  consists of the standard representations of  $SL(2, C)$  current algebra whose lowest energy states are given by principal continuous representations.

Let us introduce a convenient coordinate system  $(\phi, \gamma, \bar{\gamma})$  on  $H_3^+$ . They are related to the embedding coordinates  $(X^0, X^1, X^2, X_E^3)$  in (1) in the following way:

$$\begin{aligned}\phi &= \log(X^0 + X_E^3)/R, \\ \gamma &= \frac{X^2 + iX^1}{X^0 + X_E^3}, \\ \bar{\gamma} &= \frac{X^2 - iX^1}{X^0 + X_E^3}.\end{aligned}\tag{3}$$

A point in the coset  $SL(2, C)/SU(2)$  has its representative as a Hermitian matrix:

$$\begin{pmatrix} \gamma\bar{\gamma}e^\phi + e^{-\phi} & -\gamma e^\phi \\ -\bar{\gamma}e^\phi & e^\phi \end{pmatrix},\tag{4}$$

and the metric can be written as

$$ds^2 = R^2 (d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma}).\tag{5}$$

In this theory, there exists an important set of primary fields defined by

$$\Phi_j(x, \bar{x}; z, \bar{z}) = \frac{1 - 2j}{\pi} (e^{-\phi} + |\gamma - x|^2 e^\phi)^{-2j}.\tag{6}$$

The labels  $x, \bar{x}$  are introduced to keep track of the  $SL(2, C)$  quantum numbers.<sup>3</sup> The  $SL(2, C)$  currents act on it as

$$J^a(z)\Phi_j(x, \bar{x}; w, \bar{w}) \sim -\frac{D^a}{z - w}\Phi_j(x, \bar{x}; w, \bar{w}), \quad a = \pm, 3,\tag{7}$$

where  $D^a$  are differential operators with respect to  $x$  defined as

$$D^+ = \frac{\partial}{\partial x}, \quad D^3 = x \frac{\partial}{\partial x} + j, \quad D^- = x^2 \frac{\partial}{\partial x} + 2jx.\tag{8}$$

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<sup>3</sup>In the string theory interpretation discussed in section 3,  $(x, \bar{x})$  is identified as the location of the operator in the dual CFT on  $S^2$  on the boundary of  $H_3^+$ .

The energy momentum tensor is given by the Sugawara construction, and the worldsheet conformal weights of this operator is

$$\Delta_j = -\frac{j(j-1)}{k-2}. \quad (9)$$

When  $j = \frac{1}{2} + is$  with  $s \in \mathbf{R}$ , the operators (6) correspond to the normalizable states in the coset model. The operators with the  $SL(2, C)$  spin  $j$  and  $(1-j)$  are not independent but are related to each other by the following reflection symmetry relation,

$$\Phi_j(x, \bar{x}; z, \bar{z}) = \mathcal{R}(j) \frac{2j-1}{\pi} \int d^2 x' |x-x'|^{-4j} \Phi_{1-j}(x', \bar{x}'; z, \bar{z}), \quad (10)$$

where

$$\mathcal{R}(j) = \nu^{1-2j} \frac{\Gamma\left(1 - \frac{2j-1}{k-2}\right)}{\Gamma\left(1 + \frac{2j-1}{k-2}\right)}. \quad (11)$$

The two and three point functions of these operators have been computed in [8, 9, 10]. The two point function has the form,

$$\begin{aligned} & \langle \Phi_j(x_1, \bar{x}_1; z_1, \bar{z}_1) \Phi_{j'}(x_2, \bar{x}_2; z_2, \bar{z}_2) \rangle \\ &= \frac{1}{|z_{12}|^{4\Delta_j}} \left[ \delta^2(x_1 - x_2) \delta(j + j' - 1) + \frac{B(j)}{|x_{12}|^{4j}} \delta(j - j') \right]. \end{aligned} \quad (12)$$

The coefficient  $B(j)$  is given by

$$B(j) = \frac{k-2}{\pi} \frac{\nu^{1-2j}}{\gamma\left(\frac{2j-1}{k-2}\right)}, \quad (13)$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \quad (14)$$

The choice of  $\nu$  will not play an important role in the discussion of this paper, except for its behavior in the semi-classical limit

$$\nu \rightarrow 1 \quad (k \rightarrow \infty), \quad (15)$$

which can be deduced by comparing (10) with its classical counterpart [28].

The three point function is expressed as

$$\begin{aligned} & \langle \Phi_{j_1}(x_1, \bar{x}_1; z_1, \bar{z}_1) \Phi_{j_2}(x_2, \bar{x}_2; z_2, \bar{z}_2) \Phi_{j_3}(x_3, \bar{x}_3; z_3, \bar{z}_3) \rangle = \\ &= C(j_1, j_2, j_3) \frac{1}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)} |z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)} |z_{31}|^{2(\Delta_3+\Delta_1-\Delta_2)}} \times \\ & \quad \frac{1}{|x_{12}|^{2(j_1+j_2-j_3)} |x_{23}|^{2(j_2+j_3-j_1)} |x_{31}|^{2(j_3+j_1-j_2)}}, \end{aligned} \quad (16)$$

with the coefficient  $C(j_1, j_2, j_3)$  given by

$$C(j_1, j_2, j_3) = -\frac{G(1-j_1-j_2-j_3)G(j_3-j_1-j_2)G(j_2-j_3-j_1)G(j_1-j_2-j_3)}{2\pi^2\nu^{j_1+j_2+j_3-1}\gamma\left(\frac{k-1}{k-2}\right)G(-1)G(1-2j_1)G(1-2j_2)G(1-2j_3)}, \quad (17)$$

where

$$G(j) = (k-2)^{\frac{j(k-1-j)}{2(k-2)}}\Gamma_2(-j \mid 1, k-2)\Gamma_2(k-1+j \mid 1, k-2), \quad (18)$$

and  $\Gamma_2(x|1, \omega)$  is the Barnes double Gamma function defined by

$$\begin{aligned} & \log(\Gamma_2(x|1, \omega)) \\ &= \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial \epsilon} \left[ \sum_{n,m=0}^{\infty} (x+n+m\omega)^{-\epsilon} - \sum_{\substack{n,m=0 \\ (n,m) \neq (0,0)}} (n+m\omega)^{-\epsilon} \right]. \end{aligned} \quad (19)$$

## 2.2 Constraints on one point functions

In this subsection, we will derive functional equations satisfied by one point functions of closed string operators on a disk with boundary conditions corresponding to  $AdS_2$  branes. We will follow the discussion in [23, 24] closely, except that we use a more general ansatz for the one point function as we will explain in the next paragraph. Given the one point functions, we can find boundary states  $|B\rangle\rangle$  for the  $AdS_2$  branes. According to [13],  $AdS_2$  branes preserve one half of the current algebra symmetry because of the boundary condition on the currents,

$$J^a(z) = \bar{J}^a(\bar{z}); \quad \text{at } z = \bar{z}, \quad a = 3, +, -. \quad (20)$$

In the closed string channel, it is translated into the condition on the boundary states as

$$(J_n^a + \bar{J}_{-n}^a)|B\rangle\rangle = 0, \quad a = 3, +, -, \quad n \in \mathbf{Z}. \quad (21)$$

We start with the ansatz that the one point function is of the form

$$\begin{aligned} \langle \Phi_j(x, \bar{x}; z, \bar{z}) \rangle &= \frac{U^+(j)}{|x - \bar{x}|^{2j}|z - \bar{z}|^{2\Delta_j}} \quad \text{for } \text{Im } x > 0, \\ &= \frac{U^-(j)}{|x - \bar{x}|^{2j}|z - \bar{z}|^{2\Delta_j}} \quad \text{for } \text{Im } x < 0. \end{aligned} \quad (22)$$

The  $z$  dependence is determined from conformal invariance on the worldsheet and the  $x$  dependence is fixed by conformal invariance on the target space. The parameters  $(x, \bar{x})$  can be regarded as coordinates on the boundary of Euclidean  $AdS_3$ , which is  $S^2$ . The  $AdS_2$  brane divides  $S^2$  into half, and the upper half plane covers one patch and the lower half the other. From the point of view of the conformal field theory on the boundary, the  $AdS_2$  introduces

an one-dimensional defect on  $S^2$ , across which the two different CFT's are glued together [14]. Therefore the one point function  $\langle \Phi_j \rangle$  may have a discontinuity across  $Im\ x = 0$ . The expression (22) allows such a discontinuity since the coefficients  $U^+$  and  $U^-$  can be different.

The reflection symmetry (10) implies a relation between  $U^+$  and  $U^-$ . Taking the expectation value of (10) on both sides, we get

$$\begin{aligned} \frac{U^\pm(j)}{|x - \bar{x}|^{2j}} &= \mathcal{R}(j) \frac{2j-1}{\pi} \left( \int_{Im\ x > 0} d^2 x' U^+(1-j) \frac{|x - x'|^{-4j}}{|x' - \bar{x}'|^{2(1-j)}} + \right. \\ &\quad \left. + \int_{Im\ x < 0} d^2 x' U^-(1-j) \frac{|x - x'|^{-4j}}{|x' - \bar{x}'|^{2(1-j)}} \right), \end{aligned} \quad (23)$$

On the left hand side, we choose  $U^+$  for  $Im\ x > 0$  and  $U^-$  for  $Im\ x < 0$ . Setting  $x' = x'_1 + ix'_2$ , we can rewrite the above  $x'$ -integration as

$$\begin{aligned} &= U^+(1-j) \int_{-\infty}^{\infty} dx'_1 \int_0^{\infty} dx'_2 \frac{(x'_1{}^2 + (x'_2 - x_2)^2)^{-2j}}{|2x'_2|^{2(1-j)}} + \\ &\quad + U^-(1-j) \int_{-\infty}^{\infty} dx'_1 \int_{-\infty}^0 dx'_2 \frac{(x'_1{}^2 + (x'_2 - x_2)^2)^{-2j}}{|2x'_2|^{2(1-j)}}. \end{aligned} \quad (24)$$

By using the Euler integral

$$\int_0^1 dx\ x^{a-1} (1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad (25)$$

we see that for the case  $x_2 > 0$ , the first term vanishes and only the second term contributes. On the other hand, when  $x_2 < 0$ , the second term vanishes and only the first term contributes. We can summarize the result as

$$U^\pm(j) = \mathcal{R}(j) U^\mp(1-j). \quad (26)$$

Introducing  $f^\pm(j)$  by

$$U^\pm(j) = \Gamma\left(1 - \frac{2j-1}{k-2}\right) \nu^{\frac{1}{2}-j} f^\pm(j), \quad (27)$$

the reflection relation (26) becomes

$$f^\pm(j) = f^\mp(1-j). \quad (28)$$

To determine  $f^+(j)$ , it is useful to consider the bulk two point function of  $\Phi_j$  with the degenerate field with  $j = -1/2$ , which satisfies

$$\partial_x^2 \Phi_{-\frac{1}{2}} = 0. \quad (29)$$

It implies that there are only two terms in its operator product expansion with  $\Phi_j$  as

$$\Phi_{-\frac{1}{2}}\Phi_j \sim C_-(j)\Phi_{j-\frac{1}{2}} + C_+(j)\Phi_{j+\frac{1}{2}} + \dots, \quad (30)$$

Here the “ $\dots$ ” denote the current algebra descendants. To simplify the equations, in this subsection, we are suppressing the dependence on  $x$  and  $z$ . The coefficients  $C_\pm(h)$  have been derived in the earlier literature, but let us compute them here for completeness. Let us take a correlation function of  $\Phi_{j'}$  with both sides of (30). The left hand side gives

$$\langle \Phi_{-\frac{1}{2}}\Phi_j\Phi_{j'} \rangle = C\left(-\frac{1}{2}, j, j'\right). \quad (31)$$

Using the expression for the three point function and the identity

$$\lim_{\epsilon \rightarrow 0} \frac{G(j_1 - j_2 + \epsilon)G(j_2 - j_1 + \epsilon)}{G(-1)G(1 + 2\epsilon)} = -2\pi(k-2)\gamma\left(\frac{k-1}{k-2}\right)\delta(j_1 - j_2), \quad (32)$$

we find that (31) contains delta functions at  $j' = j \pm \frac{1}{2}$ . On the other hand, the right hand side of (30) gives

$$\langle \Phi_{-\frac{1}{2}}\Phi_j\Phi_{j'} \rangle \sim \delta\left(j' - j + \frac{1}{2}\right) C_-(j) B\left(j - \frac{1}{2}\right) + \delta\left(j' - j - \frac{1}{2}\right) C_+(j) B\left(j + \frac{1}{2}\right) + \dots. \quad (33)$$

Comparing the coefficients of the delta functions, we find

$$\begin{aligned} C_+(j) &= \nu \frac{\gamma\left(-\frac{1}{k-2}\right)}{\gamma\left(-\frac{2}{k-2}\right)}, \\ C_-(j) &= \frac{\gamma\left(-\frac{1}{k-2}\right) \gamma\left(\frac{2j-2}{k-2}\right)}{\gamma\left(-\frac{2}{k-2}\right) \gamma\left(\frac{2j-1}{k-2}\right)}. \end{aligned} \quad (34)$$

Given the coefficient  $C_\pm(j)$  for the operator product expansion of  $\Phi_{-\frac{1}{2}}\Phi_j$ , one can deduce a functional relation for  $U^+(j)$ .<sup>4</sup> Consider the following bulk two point function involving a degenerate field  $\Phi_{-1/2}$

$$\langle \Phi_{-\frac{1}{2}}(x; z) \Phi_j(x'; z') \rangle. \quad (35)$$

Using the operator product expansion derived in the above paragraph, we find

$$\begin{aligned} \langle \Phi_{-\frac{1}{2}}(x; z) \Phi_j(x'; z') \rangle &= \frac{|x' - \bar{x}'|^{-1-2j}}{|x - \bar{x}|^{-2}} \frac{|z' - \bar{z}'|^{-\frac{3u}{2}-2\Delta_j}}{|z - \bar{z}|^{-3u}} \times \\ &\quad \left( C_+(j) U^+\left(j + \frac{1}{2}\right) \mathcal{F}_+(x, x'; z, z') + C_-(j) U^+\left(j - \frac{1}{2}\right) \mathcal{F}_-(x, x'; z, z') \right). \end{aligned} \quad (36)$$

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<sup>4</sup>Here we take  $x$  to be on the upper half plane. A similar relation for  $U^-$  can be derived by considering  $x$  in the lower half plane. Since  $U^\pm$  are related to each other by the reflection relation, it is sufficient to determine conditions on  $U^+$ .



Here  $\mathcal{F}_\pm$  are four-point conformal blocks. According to [8], we have

$$\begin{aligned} \mathcal{F}_+(x, x'; z, z') &= \eta^{u(1-j)}(1-\eta)^{-\frac{u}{2}} \left( \chi F(-u, 1-2uj, 1-u(2j-1); \eta) + \right. \\ &\quad \left. + \frac{u\eta}{1+u(1-2j)} F(1-u, 1-2uj, 2-u(2j-1); \eta) \right), \end{aligned} \quad (37)$$

$$\begin{aligned} \mathcal{F}_-(x, x'; z, z') &= \eta^{uj}(1-\eta)^{-\frac{u}{2}} \left( \chi \frac{1}{2j-1} F(2u(j-1), 1-u, u(2j-1)+1; \eta) + \right. \\ &\quad \left. + F(1-u, -u, u(2j-1); \eta) \right), \end{aligned} \quad (38)$$

where  $F(a, b, c; z)$  are the hypergeometric functions  ${}_2F_1(a, b, c; z)$ ,

$$u = \frac{1}{k-2}, \quad (39)$$

and  $\chi$  and  $\eta$  are cross ratios of the target space and the worldsheet coordinates,

$$\chi = \frac{|x-x'|^2}{|x-\bar{x}'|^2}, \quad \eta = \frac{|z-z'|^2}{|z-\bar{z}'|^2}. \quad (40)$$

It was pointed out in [23] that, when the operator  $\Phi_{-\frac{1}{2}}$  approaches the boundary of the worldsheet, it overlaps only with the identity operator and the operator with  $j = -1$ . This is analogous to the situation in the Liouville theory discussed in [10]. Using this, the two point function  $\langle \Phi_{-\frac{1}{2}} \Phi_j \rangle$  can be evaluated as

$$\begin{aligned} \langle \Phi_{-\frac{1}{2}}(x; z) \Phi_j(x'; z') \rangle &= \frac{|x' - \bar{x}'|^{-1-2j} |z' - \bar{z}'|^{-\frac{3u}{2}-2\Delta_j}}{|x - \bar{x}'|^{-2} |z - \bar{z}'|^{-3u}} \times \\ &\quad (B_+(j) \mathcal{G}_+(x, x'; z, z') + B_-(j) \mathcal{G}_-(x, x'; z, z')), \end{aligned} \quad (41)$$

where

$$\begin{aligned} \mathcal{G}_+(x, x'; z, z') &= \eta^{uj}(1-\eta)^{\frac{3u}{2}} \left( (1-\chi) F(1+u, 2uj, 1+2u; 1-\eta) - \right. \\ &\quad \left. (1-\eta) \frac{2uj}{1+2u} F(1+u, 1+2uj, 2(1+u); 1-\eta) \right), \end{aligned} \quad (42)$$

$$\begin{aligned} \mathcal{G}_-(x, x'; z, z') &= \eta^{uj}(1-\eta)^{-\frac{u}{2}} \left( \frac{1-\chi}{2} F(2u(j-1), 1-u, 1-2u; 1-\eta) + \right. \\ &\quad \left. F(2u(j-1), -u, -2u; 1-\eta) \right), \end{aligned} \quad (43)$$

and  $u$  is given by (39). The conformal blocks in the two expressions, (36) and (41), are related to each other as

$$\mathcal{F}_- = \frac{\Gamma(u(2j-1))\Gamma(1+2u)}{\Gamma(2uj)\Gamma(1+u)} \mathcal{G}_- + \frac{\Gamma(u(2j-1))\Gamma(-2u)}{\Gamma(2u(j-1))\Gamma(-u)} \mathcal{G}_+, \quad (44)$$

$$\mathcal{F}_+ = \frac{\Gamma(1+u(1-2j))\Gamma(1+2u)}{\Gamma(1+2u(1-j))\Gamma(1+u)} \mathcal{G}_- - \frac{\Gamma(1+u(1-2j))\Gamma(-2u)}{\Gamma(1-2uj)\Gamma(-u)} \mathcal{G}_+. \quad (45)$$

(See appendix C for some useful identities involving the hypergeometric function.) According to [19, 23, 24],  $B_+$  is given by

$$B_+(j) = A_0 U^+(j), \quad (46)$$

where  $A_0$  is a constant that depends on the boundary condition and can be interpreted as the fusion coefficient of  $\Phi_{-1/2}$  with the boundary unit operator. Likewise,  $B_-(j)$  term can be interpreted as the contribution coming from the fusion with  $j = -1$  boundary operator. Comparing the terms dependent on  $\mathcal{G}_+$  in (36) and (41), we derive the following functional equation for  $U^+(j)$ :

$$\begin{aligned} A_0 U^+(j) = & \frac{\Gamma(-2u)}{\Gamma(-u)} \left( C_-(j) U^+ \left( j - \frac{1}{2} \right) \frac{\Gamma(u(2j-1))}{\Gamma(2u(j-1))} - \right. \\ & \left. C_+(j) U^+ \left( j + \frac{1}{2} \right) \frac{\Gamma(1+u(1-2j))}{\Gamma(1-2uj)} \right). \end{aligned} \quad (47)$$

Substituting the expression for  $C_{\pm}(j)$  given by (34), this reduces to

$$A_0 \nu^{-1/2} \frac{\Gamma\left(1 + \frac{1}{k-2}\right)}{\Gamma\left(1 + \frac{2}{k-2}\right)} f^+(j) = f^+ \left( j - \frac{1}{2} \right) - f^+ \left( j + \frac{1}{2} \right). \quad (48)$$

Given  $f^+(j)$  satisfying (48), we get  $f^-(j)$  by using (28).

It is convenient to introduce a parameter  $\Theta$  by

$$A_0 \nu^{-1/2} \frac{\Gamma\left(1 + \frac{1}{k-2}\right)}{\Gamma\left(1 + \frac{2}{k-2}\right)} = -2 \sinh \Theta. \quad (49)$$

We can regard  $\Theta$  as parametrizing the boundary condition as  $A_0$  depends on it. In fact, the semi-classical analysis in the next section shows that  $\Theta$  specifies the location of the  $AdS_2$  brane. A general solution to (48) and (28) is then a linear combination of

$$f^{\pm}(j) = \exp [\pm(\Theta + i2n\pi)(2j-1)] , \quad \exp [\pm(i\pi(2n+1) - \Theta)(2j-1)], \quad (50)$$

where  $n \in \mathbf{Z}$ .

### 3 Boundary states for $AdS_2$ branes

Among the solutions (50), we claim that the one which correctly represents the boundary condition for a single  $AdS_2$  brane is

$$f_{\Theta}^{\pm}(j) = C e^{\pm\Theta(2j-1)}, \quad (51)$$

where  $C$  is a constant independent of  $j$  but may depend on  $\Theta$  and  $k$ . From now on, we neglect this constant and it will not affect the rest of our discussion. In this section, we will provide evidences for this claim.

Given the  $f^\pm(j)$  solution (51), and therefore the one point function  $U^\pm(j)$ , the boundary state is expressed as

$$|B\rangle\rangle_\Theta = \int_{\frac{1}{2}+i\mathbf{R}^+} dj \left( \int_{Imx>0} d^2x \frac{U_\Theta^+(1-j)}{|x-\bar{x}|^{2(1-j)}} |j, x, \bar{x}\rangle\rangle_I + \int_{Imx<0} d^2x \frac{U_\Theta^-(1-j)}{|x-\bar{x}|^{2(1-j)}} |j, x, \bar{x}\rangle\rangle_I \right), \quad (52)$$

where  $|j, x, \bar{x}\rangle\rangle_I$  is an Ishibashi state built on the primary state  $|j, x, \bar{x}\rangle$ . The coefficients are chosen so that the one point functions are correctly reproduced as

$$\langle j, x, \bar{x} | B \rangle\rangle_\Theta = \lim_{z \rightarrow \infty} |z - \bar{z}|^{2\Delta_j} \langle \Phi_j(x, \bar{x}; z, \bar{z}) \rangle. \quad (53)$$

It can be verified using the two point function (12) and the reflection relation (26) that the boundary state given by (52) satisfies this condition. In the following, we will examine aspects of  $|B\rangle\rangle_\Theta$ .

### 3.1 Semi-classical analysis

Let us first study the semi-classical limit of the boundary state for the solution (51). We use the method of [29, 24] to identify the D-brane configuration corresponding to the boundary state. In the semi-classical limit ( $k \rightarrow \infty$ ), we can consider a closed string state  $|g\rangle$  localized at  $g \in H_3^+$ . Namely it is defined so that

$$\langle g | j, x, \bar{x} \rangle = \Phi_j(x, \bar{x} | g). \quad (54)$$

The overlap of the localized state  $|g\rangle$  with the boundary state  $|B\rangle\rangle_\Theta$  can then tell us about the configuration of the brane. (An analogous idea has been used in [30] to characterize boundary states for D-branes wrapping on cycles in Calabi-Yau manifolds.) In the limit  $k \rightarrow \infty$ , the overlap  $\langle g | B \rangle\rangle_\Theta$  is simplified as

$$\begin{aligned} & \lim_{k \rightarrow \infty} \langle g | B \rangle\rangle_\Theta \\ &= \int_{\frac{1}{2}+i\mathbf{R}^+} dj \left( \int_{Imx>0} d^2x \frac{f_\Theta^+(1-j)}{|x-\bar{x}|^{2(1-j)}} \Phi_j(x, \bar{x} | g) + \int_{Imx<0} d^2x \frac{f_\Theta^-(1-j)}{|x-\bar{x}|^{2(1-j)}} \Phi_j(x, \bar{x} | g) \right), \end{aligned}$$

where we used (15). First let us focus on the integral on the upper half plane,

$$\int_{\frac{1}{2}+i\mathbf{R}^+} dj \frac{2j-1}{\pi} \left( \int_{x_2>0} d^2x \frac{f_\Theta^+(1-j)}{(2x_2)^{2(1-j)}} \frac{1}{[e^\phi(x_1^2 + x_2^2 - (\gamma + \bar{\gamma})x_1 + i(\gamma - \bar{\gamma})x_2 + \gamma\bar{\gamma}) + e^{-\phi}]^{2j}} \right). \quad (55)$$

After integrating over  $x_1$  and changing integration variable  $x'_2 = e^\phi x_2$ , we get

$$\int_{\frac{1}{2}+i\mathbf{R}^+} dj \frac{2j-1}{\sqrt{\pi}2^{2-2j}} \frac{\Gamma(2j-\frac{1}{2})}{\Gamma(2j)} f_\Theta^+(1-j) \int_0^\infty dx'_2 (x'_2)^{2j-2} \left[ \left( x'_2 + \frac{i}{2}(\gamma - \bar{\gamma})e^\phi \right)^2 + 1 \right]^{\frac{1}{2}-2j}. \quad (56)$$

In terms of  $AdS_2$  coordinates defined in Appendix B, we have

$$\frac{i}{2}(\gamma - \bar{\gamma})e^\phi = -\sinh \psi. \quad (57)$$

The  $x'_2$ -integral is performed in appendix A, and the result is

$$\int_{\frac{1}{2}+i\mathbf{R}^+} dj f_\Theta^+(1-j) \frac{e^{\psi(2j-1)}}{\cosh \psi} = \int_0^\infty ds \frac{e^{2i(\psi-\Theta)s}}{\cosh \psi}, \quad (58)$$

where  $j = 1/2 + is$ . Likewise, we can perform the integral over the lower half plane. Combining the results together, we find that the overlap is given by

$$\lim_{k \rightarrow \infty} \langle g|B \rangle_\Theta = \int_0^\infty ds \frac{e^{2i(\psi-\Theta)s}}{\cosh \psi} + \int_0^\infty ds \frac{e^{-2i(\psi-\Theta)s}}{\cosh \psi} = \int_{-\infty}^\infty ds \frac{e^{2i(\psi-\Theta)s}}{\cosh \psi} \quad (59)$$

$$= \frac{1}{4\pi \cosh \psi} \delta(\psi - \Theta) = \frac{1}{4\pi} \delta(\sinh \psi - \sinh \Theta). \quad (60)$$

Thus we found that  $\langle g|B \rangle_\Theta$  has a support in the two-dimensional subspace at  $\psi = \Theta$  in  $AdS_3$ . Namely the parameter  $\Theta$  of the boundary state specifies the location of the  $AdS_2$  brane.

In this formalism, the insertion of the identity operator should reproduce the Born-Infeld action in the semi-classical regime. Using the reflection symmetry and taking the large  $k$  limit in (23), we see that

$$\langle 1 \rangle_\Theta \propto \cosh \Theta \int d^2x |x - \bar{x}|^{-2}. \quad (61)$$

The Born-Infeld action of  $AdS_2$  branes has been computed by independent methods in [13] to be

$$S_{BI} \propto \cosh \psi \int d\omega dt \cosh \omega. \quad (62)$$

If we interpret  $\int d^2x |x - \bar{x}|^{-2}$  as the volume of  $AdS_2$  branes, the two expressions agree with the identification  $\psi = \Theta$ .

The one point function is given by a linear combination of the solutions (50) to the functional equations discussed in the last section. Since we found that  $f_\Theta^\pm \sim e^{\pm\Theta(2j-1)}$  reproduces the correct semi-classical geometry of the  $AdS_2$  brane, the coefficients for all other solutions in (50) should vanish in the semi-classical limit  $k \rightarrow \infty$ . In fact we can make a stronger statement. If we assume the state  $|g\rangle$  satisfying (54) exists at finite value of  $k$ , then the  $j$ -integral to compute overlap  $\langle g|B \rangle_\Theta$  is finite only for the particular solution,  $f_\Theta^\pm(j)$ . For all other solutions in (50), the integral is divergent for  $k > 3$ . This suggests that the coefficients in front of the other solutions in (50) should vanish identically even for finite  $k$ .

### 3.2 Annulus amplitudes

Given the boundary states, the partition function on the annulus worldsheet with the boundary conditions  $\Theta_1$  and  $\Theta_2$  on the two boundary of the annulus is computed as exchanges of closed string states between  $|B\rangle\rangle_{\Theta_1}$  and  $|B\rangle\rangle_{\Theta_2}$ . If we view this in the open string channel, one should be able to express it as a sum over states of open string stretched between the two  $AdS_2$  branes at  $\Theta_1$  and  $\Theta_2$ . These open string states are normalizable. For the Euclidean  $AdS_2$  branes in Euclidean  $AdS_3$ , they belong to principal continuous representations.

We want to compute the following amplitude between two boundary states:

$$\begin{aligned} & \Theta_1 \langle\langle B | (q_c^{\frac{1}{2}})^{L_0 + \bar{L}_0 - \frac{c}{12}} | B \rangle\rangle_{\Theta_2} \\ &= \int_{\frac{1}{2} + i\mathbf{R}^+} dj \left( \int_{Imx > 0} d^2x \frac{U^+_{\Theta_1}(j) U^+_{\Theta_2}(1-j)}{|x - \bar{x}|^2} + \int_{Imx < 0} d^2x \frac{U^-_{\Theta_1}(j) U^-_{\Theta_2}(1-j)}{|x - \bar{x}|^2} \right) \frac{q_c^{\frac{s^2}{k-2}}}{\eta(q_c)^3} \\ &= \int d^2x |x - \bar{x}|^{-2} \cdot \int_0^\infty ds \frac{\cos 2(\Theta_1 - \Theta_2)s}{\sinh \frac{2\pi}{k-2}s} \frac{2\pi}{k-2} s \frac{q_c^{\frac{s^2}{k-2}}}{\eta(q_c)^3}. \end{aligned}$$

Using the fact

$$s \frac{q_c^{\frac{s^2}{k-2}}}{\eta(q_c)^3} = \frac{2\sqrt{2}}{\sqrt{k-2}} \int_0^\infty ds' \sin\left(\frac{4\pi}{k-2}ss'\right) s' \frac{q_o^{\frac{s'^2}{k-2}}}{\eta(q_o)^3}, \quad (63)$$

where

$$q_c = e^{2\pi i\tau_c}, \quad q_o = e^{2\pi i\tau_o}, \quad \tau_o = -\frac{1}{\tau_c}, \quad (64)$$

we can go to the open string channel:

$$\begin{aligned} &= \int d^2x |x - \bar{x}|^{-2} \cdot \frac{4\sqrt{2}\pi}{(k-2)^{3/2}} \int_0^\infty ds' \left( \int_0^\infty ds \frac{\cos(2\Theta_{12}s) \sin\left(\frac{4\pi}{k-2}ss'\right)}{\sinh \frac{2\pi}{k-2}s} \right) s' \frac{q_o^{\frac{s'^2}{k-2}}}{\eta(q_o)^3} \\ &= \int d^2x |x - \bar{x}|^{-2} \cdot \frac{\pi}{\sqrt{2}\sqrt{k-2}} \int_0^\infty ds' \frac{2s' \sinh 2\pi s'}{\cosh((k-2)\Theta_{12}) + \cosh(2\pi s')} \frac{q_o^{\frac{s'^2}{k-2}}}{\eta(q_o)^3}, \end{aligned}$$

where  $\Theta_{12} = \Theta_1 - \Theta_2$ . Obviously, the overall factor involving the  $x$ -integral is divergent. We can identify it as the volume divergence coming from the fact that the  $AdS_2$  brane is non-compact. This divergence also reflects the fact that normalizable states in the open string sector belong to principal continuous representations which are infinite dimensional.<sup>5</sup>

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<sup>5</sup>Here we have focused on the part of the annulus amplitude that scales as the volume of the  $AdS_2$  brane. After the completion of the manuscript, we were informed by B. Ponsot, V. Schomerus, and J. Teschner that, under a certain regularization of the volume divergence, one finds a finite additive term to the annulus partition function with nontrivial dependence on  $s$  and  $\Theta$ . This would add corrections to the spectral density (65) that do not scale as the volume of the  $AdS_2$  brane.

If we define  $\rho(s)$  as the density of states in open string Hilbert space belonging to the current algebra representation whose ground states are in principal continuous representation with  $j = \frac{1}{2} + is$ , the above result shows that it is given by

$$\rho(s) \propto \frac{s \sinh 2\pi s}{\cosh((k-2)\Theta_{12}) + \cosh(2\pi s)}. \quad (65)$$

The spectral density is real and non-negative as it should be. For the case  $\Theta_1 = \Theta_2$ ,  $\Theta_1$  dependence completely disappears. Note, however, we have neglected the overall constant  $C$  in the one point function (51), which can be  $\Theta$ -dependent but is independent of  $s$ .

### 3.3 One loop free energy at finite temperature

In this subsection, we consider  $AdS_2$  branes in finite-temperature  $AdS_3$  and compute the partition function by using the boundary states we have constructed. For the boundary state with  $\Theta = 0$ , we can directly compare the result with that of appendix A in [15].

Finite-temperature  $AdS_3$  is given by identifying the Euclidean time  $t_E = it$  in the target space as,

$$t_E \sim t_E + \beta. \quad (66)$$

It induces identification of boundary coordinates as well,

$$|x| \sim |x|e^\beta. \quad (67)$$

The thermal identification introduces new sectors of closed strings winding around the compact time direction [25]. As shown in [1], the winding sectors are generated by the spectral flow

$$g \rightarrow e^{\frac{i}{2}w_R x^+ \sigma_2} g e^{\frac{i}{2}w_L x^- \sigma_2}, \quad (68)$$

where  $\sigma_1, \sigma_2$ , and  $\sigma_3$  are the Pauli matrices. If we choose  $w_R = -w_L = -iw\beta/2\pi$  with  $w \in \mathbf{Z}$ , the action generates the following change in the string coordinates in Euclidean  $AdS_3$ ,

$$\begin{aligned} t_E &\sim t_E + \frac{w\beta}{2\pi}\sigma, \\ \theta &\sim \theta - \frac{w\beta}{2\pi}\tau, \\ \rho &\sim \rho. \end{aligned} \quad (69)$$

The string worldsheet remains periodic in  $\sigma \rightarrow \sigma + 2\pi$ , modulo the identification (66). This induces the following transformation on the Virasoro generator,

$$\begin{aligned} L_0 &\rightarrow L_0 + iw \frac{\beta}{2\pi} J_0^3 + kw^2 \frac{\beta^2}{16\pi^2}, \\ \bar{L}_0 &\rightarrow \bar{L}_0 - iw \frac{\beta}{2\pi} \bar{J}_0^3 + kw^2 \frac{\beta^2}{16\pi^2}. \end{aligned} \quad (70)$$

Correspondingly the boundary states include all the winding sectors:

$$|B; \beta\rangle\rangle_{\Theta} = \sum_w |B; \beta\rangle\rangle_{\Theta, w}. \quad (71)$$

Here  $|B; \beta\rangle\rangle_{\Theta, w=0}$  is the boundary state given by (52), except that the  $x$  integral is restricted in the range

$$e^{-\beta} \leq |x| \leq 1, \quad (72)$$

which is the fundamental domain of the identification (67). The other states  $|B; \beta\rangle\rangle_{\Theta, w}$  are given by performing the spectral flow (70). The amplitude we want to compute then is

$$\Theta_1 \langle\langle B; \beta | (q_c^{\frac{1}{2}})^{L_0 + \bar{L}_0 - \frac{c}{12}} | B; \beta \rangle\rangle_{\Theta_2} = \sum_{w=-\infty}^{\infty} \Theta_{1, w} \langle\langle B; \beta | (q_c^{\frac{1}{2}})^{L_0 + \bar{L}_0 - \frac{c}{12}} | B; \beta \rangle\rangle_{\Theta_{2, w}}. \quad (73)$$

The overlap in the  $w$  winding sector can be expressed as

$$\begin{aligned} & \Theta_{1, w} \langle\langle B; \beta | (q_c^{\frac{1}{2}})^{L_0 + \bar{L}_0 - \frac{c}{12}} | B; \beta \rangle\rangle_{\Theta_{2, w}} \\ &= q_c^{\frac{k\beta^2}{16\pi^2} w^2} \Theta_{1, w=0} \langle\langle B; \beta | (q_c^{\frac{1}{2}})^{L_0 + \bar{L}_0 - \frac{c}{12}} e^{\frac{\beta\tau_c}{2} w (-J_0^3 + \bar{J}_0^3)} | B; \beta \rangle\rangle_{\Theta_{2, w=0}}, \end{aligned} \quad (74)$$

where  $\tau_c = it_c$  and  $\phi_0 = -\beta t_c w/2$ . Substituting (52) into this, we find that the right-hand side is expressed as an integral over  $x$  in the range (72). The integration domain is divided into four regions, depending on the signs of  $Imx$  and  $Im(e^{i\phi_0}x)$ . Combining them together, we find

$$\begin{aligned} & -2q_c^{\frac{k\beta^2}{16\pi^2} w^2} \int_{\frac{1}{2} + i\mathbf{R}^+} dj \left( \int_{Imx > 0, Im(xe^{i\phi_0}) > 0, e^{-\beta} < |x| < 1} d^2x |x - \bar{x}|^{-2j} |xe^{i\phi_0} - \bar{x}e^{-i\phi_0}|^{-2(1-j)} \right) \times \\ & \left( U_{\Theta_1}^+(j) U_{\Theta_2}^+(1-j) + U_{\Theta_1}^+(j) U_{\Theta_2}^-(1-j) + U_{\Theta_1}^-(j) U_{\Theta_2}^+(1-j) + U_{\Theta_1}^-(j) U_{\Theta_2}^-(1-j) \right) \times \\ & \sin \phi_0 \frac{q_c^{s^2/(k-2)}}{\vartheta_{11}(\frac{\phi_0}{\pi} | it_c)}. \end{aligned}$$

The  $x$ -integral is performed in appendix A,

$$(x\text{-integral}) = \frac{\pi\beta}{|\sin \phi_0|} \delta(s). \quad (75)$$

Thus the overlap in the winding number  $w$  sector is given by

$$\Theta_{1, w} \langle\langle B; \beta | (q_c^{\frac{1}{2}})^{L_0 + \bar{L}_0 - \frac{c}{12}} | B; \beta \rangle\rangle_{\Theta_{2, w}} \propto \beta \frac{e^{-\frac{kt_c\beta^2}{8\pi} w^2}}{|\vartheta_{11}(\frac{\beta t_c w}{2\pi} | it_c)|}. \quad (76)$$

Altogether we have,

$$\Theta_1 \langle\langle B; \beta | (q_c^{\frac{1}{2}})^{L_0 + \bar{L}_0 - \frac{c}{12}} | B; \beta \rangle\rangle_{\Theta_2} \propto \sum_{w=-\infty}^{\infty} \beta \frac{e^{-\frac{kt_c\beta^2}{8\pi} w^2}}{|\vartheta_{11}(\frac{\beta t_c w}{2\pi} | it_c)|}. \quad (77)$$

Note that the dependence on  $\Theta_1$  and  $\Theta_2$  has disappeared after the  $j$ -integral. In fact, the expression (77) agrees precisely with the result of [15] in the case of  $\Theta_1 = \Theta_2 = 0$ , *i.e.* when the branes carry no fundamental string charge. There the annulus amplitude is evaluated exactly using the iterative Gaussian integral method developed in [27, 2]. To compute the one loop free energy, we need to multiply the partition functions of the ghost sector and the internal CFT and to integrate the result over the moduli space of the annulus worldsheet. The method has been explained in detail in the appendix of [15], and we do not repeat it here.

We have found that the boundary state  $|B; \beta\rangle_\Theta$  precisely reproduces the one loop free energy computed in [15] when  $\Theta_1 = \Theta_2 = 0$ . It turned out that the partition function does not depend on  $\Theta_1$  or  $\Theta_2$  at all. The reason for this is unclear and deserves a closer inspection.

## 4 Conclusion

In this paper, we constructed the boundary states for  $AdS_2$  branes in  $AdS_3$ , following [23, 24] but using the different ansatz. The boundary states are expressed as linear combinations of the Ishibashi states as in (52), where the one point functions  $U_\Theta^\pm(j)$  are given by,

$$U_\Theta^\pm(j) = \Gamma\left(1 - \frac{2j-1}{k-2}\right) \nu^{\frac{1}{2}-j} e^{\pm\Theta(2j-1)}, \quad (78)$$

modulo factors independent of  $j$ . In the semi-classical approximation, the location of the brane is given by  $\psi = \Theta$  in the  $AdS_2$  coordinates defined in Appendix B.

From the point of view of the boundary conformal field theories, the  $AdS_2$  branes create defects which connect different conformal field theories while preserving at least one Virasoro algebra. Since the boundary states allow study of the  $AdS_2$  branes beyond the supergravity approximation, it would be interesting to use them to explore the correspondence further.

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## A Some useful integrals

### A.1 Integral used in the semi-classical approximation

We start with the following integral formula,

$$\int_0^\infty dx \frac{x^n}{(ax^2 + 2bx + c)^{n+3/2}} = \frac{n!}{(2n+1)!!\sqrt{c}(\sqrt{ac} + b)^{n+1}}, \quad (79)$$

where  $a \geq 0, c > 0, b > -\sqrt{ac}$ . We want to integrate

$$\int_0^\infty dx'_2 (x'_2)^{2j-2} (x'^2_2 - 2 \sinh \psi x'_2 + \cosh^2 \psi)^{\frac{1}{2}-2j}. \quad (80)$$

Using  $n = 2j - 2$ ,  $a = 1$ ,  $c = \cosh^2 \psi$  and  $b = -\sinh \psi$ , we see the restrictions are clearly satisfied for all values of  $\psi \in \mathbf{R}$ . Using the fact that  $\Gamma(1/2 + n) = \sqrt{\pi} 2^{-n} (2n-1)!!$ , we have

$$\int_0^\infty dx'_2 (x'_2)^{2j-2} (x'^2_2 - 2 \sinh \psi x'_2 + \cosh^2 \psi)^{\frac{1}{2}-2j} = \frac{2^{2-2j} \Gamma(2j-1) \sqrt{\pi} e^{\psi(2j-1)}}{\Gamma(2j - \frac{1}{2}) \cosh \psi}. \quad (81)$$

### A.2 Integral used for finite temperature free energy

The integral we want to compute is:

$$\int_{\text{Im} x > 0, \text{Im}(xe^{i\phi_0}) > 0, e^{-\beta} < |x| < 1} d^2 x |x - \bar{x}|^{-2j} |xe^{i\phi_0} - \bar{x}e^{-i\phi_0}|^{-2(1-j)}. \quad (82)$$

Use polar coordinates such that  $x = re^{i\theta}$ . Then we get,

$$\begin{aligned} &= \left( \int_{e^{-\beta}}^1 \frac{dr}{r} \right) \int_0^{\pi-\phi_0} d\theta |\sin \theta \sin(\theta + \phi_0)|^{-1} \exp \left( 2is \ln \left| \frac{\sin(\theta + \phi_0)}{\sin \theta} \right| \right) \\ &= \frac{\beta}{2 \sin \phi_0} \int_{-\infty}^\infty d\tilde{s} e^{-is\tilde{s}} \frac{\pi}{|\sin \phi_0|} \end{aligned} \quad (83)$$

$$= \frac{\pi\beta}{|\sin \phi_0|} \delta(s), \quad (84)$$

where, in the second line, we changed the variable  $\theta$  to  $\tilde{s}$  defined by

$$\tilde{s} = 2 \ln \left| \frac{\sin(\theta + \phi_0)}{\sin \theta} \right|. \quad (85)$$

## B Coordinate Systems for $AdS_3$

The space  $AdS_3$  is defined as the hyperboloid

$$(X^0)^2 - (X^1)^2 - (X^2)^2 + (X^3)^2 = R^2, \quad (86)$$

embedded in  $\mathbf{R}^{2,2}$ . The metric

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 - (dX^3)^2 \quad (87)$$

on  $\mathbf{R}^{2,2}$  induces a metric of constant negative curvature on  $AdS_3$ . The quantity  $R$  that appears in (86) is the *anti-de Sitter radius*; for convenience, we set  $R = 1$ . In addition, to avoid closed time-like curves, we work not with the hyperboloid (86) itself, but with its universal cover.

The two coordinate systems we use most extensively are *global coordinates* and  $AdS_2$  *coordinates*. The global coordinates  $(\rho, \theta, \tau)$  are defined by

$$X^0 + iX^3 = \cosh \rho e^{it}, \quad X^1 + iX^2 = -\sinh \rho e^{-i\theta}. \quad (88)$$

The range of the radial coordinate  $\rho$  is  $0 \leq \rho < \infty$ ; the angular coordinate  $\theta$  ranges over  $0 \leq \theta < 2\pi$ ; and the global time coordinate  $t$  may be any real number. The  $AdS_3$  metric in global coordinates is

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2. \quad (89)$$

The  $AdS_2$  coordinate system is convenient to use when considering  $AdS_2$  branes in  $AdS_3$ . They are defined by

$$X^1 = \cosh \psi \sinh \omega, \quad X^2 = \sinh \psi, \quad X^0 + iX^3 = \cosh \psi \cosh \omega e^{it}. \quad (90)$$

All three  $AdS_2$  coordinates range over the entire real line. In this parametrization, the fixed  $\psi$  slices have the geometry of  $AdS_2$ . The  $AdS_3$  metric in  $AdS_2$  coordinates takes the form

$$ds^2 = d\psi^2 + \cosh^2 \psi (-\cosh^2 \omega dt^2 + d\omega^2); \quad (91)$$

the quantity in parentheses is the metric of the  $AdS_2$  subspace at fixed  $\psi$ . The transformation between global and  $AdS_2$  coordinates is

$$\sinh \psi = \sin \theta \sinh \rho, \quad \cosh \psi \sinh \omega = -\cos \theta \sinh \rho. \quad (92)$$

The global time  $t$  is the same in both coordinate systems.

The space  $AdS_3$  is the group manifold of the group  $SL(2, R)$ . A point in  $AdS_3$  is given by the  $SL(2, R)$  matrix

$$g = \begin{pmatrix} X^0 + X^1 & X^2 + X^3 \\ X^2 - X^3 & X^0 - X^1 \end{pmatrix}. \quad (93)$$

In the global coordinate system,

$$g = \begin{pmatrix} \cos t \cosh \rho - \cos \theta \sinh \rho & \sin t \cosh \rho + \sin \theta \sinh \rho \\ -\sin t \cosh \rho + \sin \theta \sinh \rho & \cos t \cosh \rho + \cos \theta \sinh \rho \end{pmatrix}. \quad (94)$$

In this paper, we work mostly in the Euclidean rotation of this geometry, which is given by  $t \rightarrow t_E = it$  in the above.

## C Some useful formulae

In this appendix, we list some useful formulae involving the hypergeometric function and the gamma function.

The hypergeometric function,  ${}_2F_1(\alpha, \beta, \gamma; z)$ , enjoys following useful identities:

$${}_2F_1(\alpha, \beta, \gamma; z) = (1 - z)^{\gamma - \alpha - \beta} {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma; z). \quad (95)$$

The Gauss recursion formulae are given by

$$\gamma {}_2F_1(\alpha, \beta, \gamma; z) + (\beta - \gamma) {}_2F_1(\alpha + 1, \beta, \gamma + 1; z) - \quad (96)$$

$$\beta(1 - z) {}_2F_1(\alpha + 1, \beta + 1, \gamma + 1; z) = 0,$$

$$\gamma {}_2F_1(\alpha, \beta, \gamma; z) - \gamma {}_2F_1(\alpha + 1, \beta, \gamma; z) + \beta z {}_2F_1(\alpha + 1, \beta + 1, \gamma + 1; z) = 0, \quad (97)$$

$$\gamma {}_2F_1(\alpha, \beta, \gamma; z) - (\gamma - \beta) {}_2F_1(\alpha, \beta, \gamma + 1; z) - \beta {}_2F_1(\alpha, \beta + 1, \gamma + 1; z) = 0. \quad (98)$$

Under  $z \rightarrow 1 - z$ , we have

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma; 1 - z) &= \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} {}_2F_1(\alpha, \beta, 1 - \gamma + \alpha + \beta; z) + \\ & z^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} {}_2F_1(\gamma - \alpha, \gamma - \beta, 1 + \gamma - \alpha - \beta; z). \end{aligned} \quad (99)$$

Lastly, under  $z \rightarrow 1/z$ , we get

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma; z) &= \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} \left(-\frac{1}{z}\right)^\alpha {}_2F_1\left(\alpha, 1 + \alpha - \gamma, 1 + \alpha - \beta; \frac{1}{z}\right) + \\ & \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} \left(-\frac{1}{z}\right)^\beta {}_2F_1\left(\beta, 1 + \beta - \gamma, 1 + \beta - \alpha; \frac{1}{z}\right). \end{aligned} \quad (100)$$

We list some useful identities involving the gamma,  $\Gamma(z)$ , function:

$$\Gamma(1 + z) = z\Gamma(z), \quad (101)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad (102)$$

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad (103)$$

$$\Gamma(1+ix)\Gamma(1-ix) = \frac{\pi x}{\sinh(\pi x)}, \quad x \in \mathbf{R} \quad (104)$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right). \quad (105)$$

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